

A SURFACE CONTAINING A LINE AND A CIRCLE THROUGH EACH POINT IS A QUADRIC

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ABSTRACT. We prove that a surface in 3-dimensional Euclidean space containing a line and a circle through each point is a quadric. We also give some particular results on the classification of surfaces containing several circles through each point.

Keywords: ruled surface, circular surface, circle, cyclide, conic bundle.

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1. INTRODUCTION

Surfaces generated by simplest curves (lines and circles) are popular subject in pure mathematics and have applications to design and architecture [12, 2]. If a surface contains two such curves through each point then we get a mesh on the surface. Famous examples of such meshes are V. G. Shukhov's hyperboloid structures. A natural question is which other surfaces can be constructed from straight and circular beams.

It is well-known that a surface containing two lines through each point (*doubly ruled surface*) must be a quadric. In this paper we show that a smooth surface containing both a line and a circle through each point still must be a quadric; see Figure 1 to the left. By a *smooth surface* we mean the image of an injective C^∞ map $\mathbb{R}^2 \rightarrow \mathbb{R}^3$ with nowhere vanishing differential.

Theorem 1.1. *If through each point of a smooth surface in \mathbb{R}^3 one can draw both a straight line segment and a circular arc transversal to each other and fully contained in the surface (and continuously depending on the point) then the surface is a piece of either a one-sheeted hyperboloid, or a quadratic cone, or an elliptic cylinder, or a plane.*

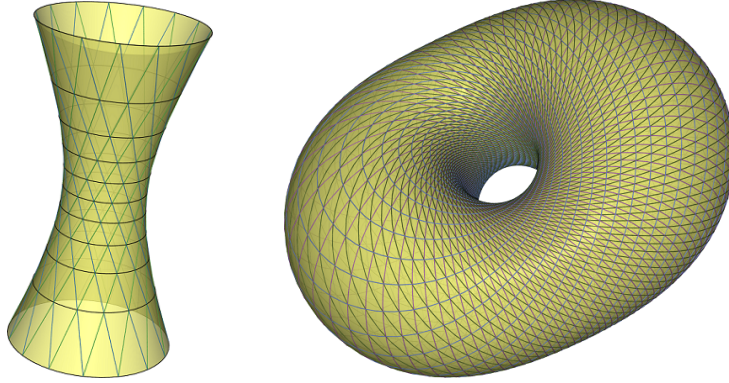


FIGURE 1. Left: A one-sheeted hyperboloid contains both a line and a circle through each point. To find all surfaces with this property (Theorem 1.1), we prove that the planes of the generating circles are parallel (Lemma 2.1) and intersect the surface only at the points of the circles (Lemma 2.6). Right: A cyclide contains several circles through each point. Finding all surfaces with this property is a challenging open problem. So far we prove that a surface with 4 circles through each point is a cyclide (Theorem 3.4).

In what follows a line (circle) continuously depending on a real parameter is called a *family* of lines (circles).

Although the proof is rather elementary, Theorem 1.1 is more tricky than the classical description of doubly ruled surfaces. Let us illustrate the difference. First, the classical result does not really require 2 lines through *each* point: a surface covered by 1 family of lines and containing just 3

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more lines intersecting them all must already be a quadric. Second, the classical result remains true in *complex* 3-space. However, similar generalizations of Theorem 1.1 are far from being true; see Examples 3.6–3.9 below.

The next natural problem, which seems to be still open (and is going to be studied in detail in a subsequent publication), is to describe all surfaces containing several circles through each point.

An example of such surface is a *cyclide*, i.e., the surface given by the equation of the form

$$(1) \quad a(x^2 + y^2 + z^2)^2 + (x^2 + y^2 + z^2)(bx + cy + dz) + Q(x, y, z) = 0,$$

where a, b, c, d are constants and $Q(x, y, z)$ is a polynomial of degree at most 2; see Figure 1 to the right. Such a surface is also called a *Darboux cyclide*, not to be confused with a *Dupin cyclide* being a particular case. This class of surfaces appears in different branches of mathematics [9, 10, 8]. An introduction to cyclides and circles on them can be found in the work of Pottmann et al. [14]. Any cyclide (besides some degenerate cases) contains at least 2 circles through each point [4, 14]. Conversely, a surface containing 2 cospherical or 2 orthogonal circles through each point must be a cyclide; see Theorem 3.5 below (taken from [4]) and [6, Theorems 1 and 2]. However, this is not true without the assumption of either cosphericity or orthogonality; see Example 3.10 below.

A torus is an example of a cyclide with 4 circles through each point: a meridian, a parallel, and 2 Villarceau circles. There are cyclides with 6 circles through each point [1, 14]. We give a short proof of the Takeuchi theorem [18] stating that a surface with 7 circles through each point must be a sphere (Theorem 3.1 below), and we generalize an old Darboux result that a surface with sufficiently many circles through each point is a cyclide (Theorem 3.4 below).

Further generalizations concern *conic bundles*, in particular, surfaces containing a conic through each point. Surfaces containing both a line and a conic through each point were classified by Brauner [3]. Such a surface has degree at most 4. Notice that it is much more difficult to deduce Theorem 1.1 from this classification than to prove Theorem 1.1 itself. Surfaces containing several conics through each point were classified by Schicho [15]. Such surfaces have degree at most 8 and admit a biquadratic rational parametrization. *Webs* of circles and conics are discussed in [14, 19, 16].

The paper is organized as follows. In Section 2 we prove Theorem 1.1. In Section 3 we state and prove some related results mentioned above.

2. PROOFS

Throughout this section we work over the field of complex numbers except otherwise is explicitly indicated. Denote by \mathbb{P}^3 the 3-dimensional complex projective space with homogeneous coordinates $x : y : z : w$. The *infinitely distant plane* is the plane $w = 0$. The *absolute conic* is given by the equations $x^2 + y^2 + z^2 = 0$, $w = 0$. A (*nondegenerate*) *complex circle* is an irreducible conic in \mathbb{P}^3 having two distinct common points with the absolute conic. Clearly, a circle in \mathbb{R}^3 is a subset of a complex circle.

The set of projective lines in \mathbb{P}^3 can be naturally identified with the *Plücker quadric* $Gr(2, 4)$ in \mathbb{P}^5 : the line passing through points $x_1 : y_1 : z_1 : w_1$ and $x_2 : y_2 : z_2 : w_2$ is identified with the point $x_1y_2 - x_2y_1 : x_1z_2 - x_2z_1 : x_1w_2 - x_2w_1 : y_1z_2 - y_2z_1 : y_1w_2 - y_2w_1 : z_1w_2 - z_2w_1$.

Let $\Phi \subset \mathbb{R}^3$ be a surface covered by a family of real line segments and a family of real circular arcs simultaneously. The complex lines and complex circles containing the members of these families are called *generating lines* and *generating circles*, respectively. Hereafter assume that $\Phi \subset \mathbb{R}^3$ is not a plane.

Lemma 2.1. *The planes of the generating circles are parallel to each other.*

To prove the lemma, we need several auxiliary claims. The first one is essentially known.

Claim 2.2. (See [5, Example 8.1]) *Let $\gamma_1, \gamma_2, \gamma_3 \subset \mathbb{P}^3$ be pairwise distinct irreducible algebraic curves.*

- (1) *The set $J(\gamma_1) \subset Gr(2, 4)$ of all the lines passing through the curve γ_1 is an algebraic subset of $Gr(2, 4)$.*
- (2) *The set $J(\gamma_1, \gamma_2) \subset Gr(2, 4)$ of all the lines passing through each of the curves γ_1, γ_2 but not passing through their intersection $\gamma_1 \cap \gamma_2$ is a piece of a 2-dimensional algebraic surface in $Gr(2, 4)$.*
- (3) *The union $J(\gamma_1, \gamma_2, \gamma_3) \subset \mathbb{P}^3$ of all the lines passing through each of the curves $\gamma_1, \gamma_2, \gamma_3$ but not passing through their pairwise intersections is a piece of an algebraic surface in \mathbb{P}^3 .*

Remark. Moreover, one can check that $J(\gamma_1, \gamma_2)$ and $J(\gamma_1, \gamma_2, \gamma_3)$ are quasi-projective varieties.

Proof. (1) The set S of pairs $(P, \lambda) \in \mathbb{P}^3 \times Gr(2, 4)$ such that the point P belongs to the line λ is algebraic. The set $J(\gamma_1)$ is the image of the projection $S \cap (\gamma_1 \times Gr(2, 4)) \rightarrow Gr(2, 4)$ and by [5, Theorem 3.12] is also algebraic.

(2) Consider the polynomial map $(\gamma_1 - \gamma_1 \cap \gamma_2) \times (\gamma_2 - \gamma_1 \cap \gamma_2) \rightarrow Gr(2, 4)$ taking a pair of points (P, Q) to the line passing through P and Q . Then $J(\gamma_1, \gamma_2)$ is the image of this map. The map $(\gamma_1 - \gamma_1 \cap \gamma_2) \times (\gamma_2 - \gamma_1 \cap \gamma_2) \rightarrow J(\gamma_1, \gamma_2)$ is a finite covering by a piece of a 2-dimensional irreducible surface. Thus $J(\gamma_1, \gamma_2)$ is a piece of a 2-dimensional irreducible algebraic surface $\bar{J}(\gamma_1, \gamma_2) \subset Gr(2, 4)$.

(3) By (2) the set $\bigcap_{i \neq j} J(\gamma_i, \gamma_j)$ is a (possibly empty) piece of the algebraic set $\bigcap_{i \neq j} \bar{J}(\gamma_i, \gamma_j)$. Since for each $i \neq j$ the surface $\bar{J}(\gamma_i, \gamma_j)$ is 2-dimensional and irreducible it follows that $\dim \bigcap_{i \neq j} \bar{J}(\gamma_i, \gamma_j) \leq 2$. If $\dim \bigcap_{i \neq j} \bar{J}(\gamma_i, \gamma_j) = 2$ then $\bar{J}(\gamma_1, \gamma_2) = \bar{J}(\gamma_2, \gamma_3) = \bar{J}(\gamma_3, \gamma_1)$ and by [7, Theorem 1, $n = 3$] it follows that $J(\gamma_1, \gamma_2, \gamma_3)$ is a piece of a plane. If $\dim \bigcap_{i \neq j} \bar{J}(\gamma_i, \gamma_j) \leq 1$ then $J(\gamma_1, \gamma_2, \gamma_3)$ is a piece of an algebraic surface as the image of the projection $S \cap (\mathbb{P}^3 \times \bigcap_{i \neq j} J(\gamma_i, \gamma_j)) \rightarrow \mathbb{P}^3$. \square

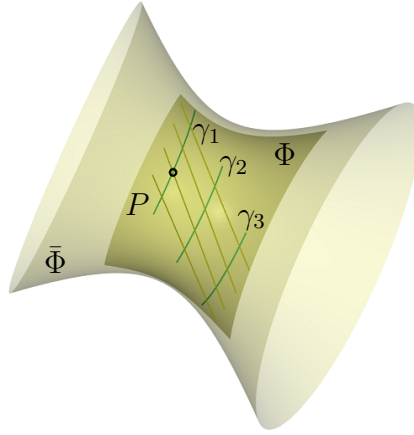


FIGURE 2. To the proof of Claim 2.3.

Claim 2.3. (Cf. [15, Theorems 1 and 2]) *The surface $\Phi \subset \mathbb{R}^3$ is contained in an irreducible ruled algebraic surface $\bar{\Phi} \subset \mathbb{P}^3$. The family of generating lines is a piece of an irreducible algebraic curve in $Gr(2, 4)$.*

Proof. Take a point $P \in \Phi$; see Figure 2. Draw a circular arc $\gamma_1 \subset \Phi$ through the point P . Draw the line segments in Φ from our continuous family through each point of γ_1 . Since the drawn segments are transversal to γ_1 , the circular arcs contained in Φ form a continuous family, and Φ is smooth it follows that there are arcs $\gamma_2, \gamma_3 \subset \Phi$ (sufficiently close to γ_1) which intersect each of the drawn segments. Let $\bar{\gamma}_1, \bar{\gamma}_2, \bar{\gamma}_3$ be the complex circles in \mathbb{P}^3 containing the arcs $\gamma_1, \gamma_2, \gamma_3$. Let $\Phi' \subset \Phi$ be the union of those drawn segments, whose lines do not pass through the intersections $\bar{\gamma}_1 \cap \bar{\gamma}_2$, $\bar{\gamma}_2 \cap \bar{\gamma}_3$, $\bar{\gamma}_3 \cap \bar{\gamma}_1$. By construction $\Phi' \subset J(\bar{\gamma}_1, \bar{\gamma}_2, \bar{\gamma}_3)$. Thus by Claim 2.2(3) the collar Φ' is a piece of an algebraic surface. Take $\bar{\Phi}$ to be an irreducible component of the surface containing a closed 2-dimensional subset of the initial surface Φ including the point P . (If there are no such components, e.g., $\Phi' = \emptyset$, then the drawn segments sufficiently close to the point P form a quadratic cone with vertex at one of the intersection points of the circles $\bar{\gamma}_1, \bar{\gamma}_2, \bar{\gamma}_3$; in this case set $\bar{\Phi}$ to be this cone.) The algebraic surface $\bar{\Phi} \subset \mathbb{P}^3$ does not depend on the point P because the smooth surface $\Phi \subset \mathbb{R}^3$ cannot jump from one irreducible algebraic surface to another.

By Claim 2.2(2) the lines containing the drawn segments form a piece of the algebraic set $\bigcap_{i \neq j} \bar{J}(\gamma_i, \gamma_j)$. Since Φ is not a plane it follows that the latter set is an algebraic curve [7, Theorem 1, $n = 3$]. Take $\alpha \subset Gr(2, 4)$ to be an irreducible component of this curve containing the lines sufficiently close to the point P . Clearly, the union of the lines of the whole curve α covers $\bar{\Phi}$, i.e., $\bar{\Phi}$ is ruled. It remains to show that the curve α does not depend on the choice of the point P . Indeed, assume that the generating lines through a neighborhood of another point P' form a curve $\alpha' \subset Gr(2, 4)$ distinct from α . Then $\bar{\Phi}$ is doubly ruled and hence it is a quadric. Thus $\alpha \cap \alpha' = \emptyset$ and hence the generating lines cannot form a continuous family. This contradiction proves the claim. \square

Hereafter any line belonging to the irreducible algebraic curve in $Gr(2, 4)$ containing the generating lines is also called a *generating line*. No confusion will arise from this.

Claim 2.4. (Cf. [11, Lemma 1.3]) *If $\gamma \subset \bar{\Phi}$ is an irreducible algebraic curve distinct from a generating line then each generating line intersects γ .*

Proof. Since $\bar{\Phi} \subset \mathbb{P}^3$ is ruled it follows that there is a generating line through each point of γ . Thus infinitely many generating lines belong to $J(\gamma)$. By Claim 2.2(1) the set $J(\gamma)$ is algebraic, hence the whole irreducible algebraic curve in $Gr(2, 4)$ formed by generating lines is contained in $J(\gamma)$. \square

Claim 2.5. *The surface $\bar{\Phi} \subset \mathbb{P}^3$ does not contain the absolute conic.*

Proof. Assume that $\bar{\Phi}$ contains the absolute conic γ . Then by Claim 2.4 all the generating lines intersect γ . Since γ has no real points it follows that there are no real generating lines (except infinitely distant). This contradiction proves the claim. \square

Proof of Lemma 2.1. By Claims 2.3 and 2.5 the intersection of the surface $\bar{\Phi} \subset \mathbb{P}^3$ with the absolute conic is a finite set I . The plane of each generating circle intersects the infinitely distant plane by a line joining two points of the set I . Since the set I is finite and the family of generating circles is continuous it follows that all these lines coincide, that is, all the planes of the circles are parallel. \square

Lemma 2.6. *There are infinitely many generating circles γ such that the projective plane $\Pi \subset \mathbb{P}^3$ of γ intersects the surface $\bar{\Phi} \subset \mathbb{P}^3$ only at the points of γ .*

To prove the lemma, we need the following auxiliary claim.

Claim 2.7. *The projective planes $\Pi \subset \mathbb{P}^3$ of infinitely many generating circles γ do not contain generating lines.*

Proof. Assume that the projective planes of only finitely many generating circles do not contain generating lines. Thus the projective planes $\Pi \subset \mathbb{P}^3$ of infinitely many generating circles γ contain generating lines λ_γ . By Lemma 2.1 all the projective planes Π intersect the absolute conic by the same 2-point set $I = \{P, Q\}$. It suffices to consider the following 3 cases.

Case 1: *For some γ we have $\lambda_\gamma \cap I = \emptyset$.* Take a generating circle $\gamma' \not\subset \Pi$. Then $\Pi \cap \gamma' = I$ by Lemma 2.1. Then by Claim 2.4 we have $\emptyset \neq \lambda_\gamma \cap \gamma' \subset \Pi \cap \gamma' = I = \emptyset$, a contradiction.

Case 2: *For infinitely many γ the intersection $\lambda_\gamma \cap I$ consists of a single point.* All the lines λ_γ with this property are pairwise distinct because by Lemma 2.1 they are contained in the planes through I . We get infinitely many generating lines intersecting I . Thus by Claim 2.3 each generating line must intersect I . Then the generating lines through each point of a generating circle γ must intersect I , a contradiction.

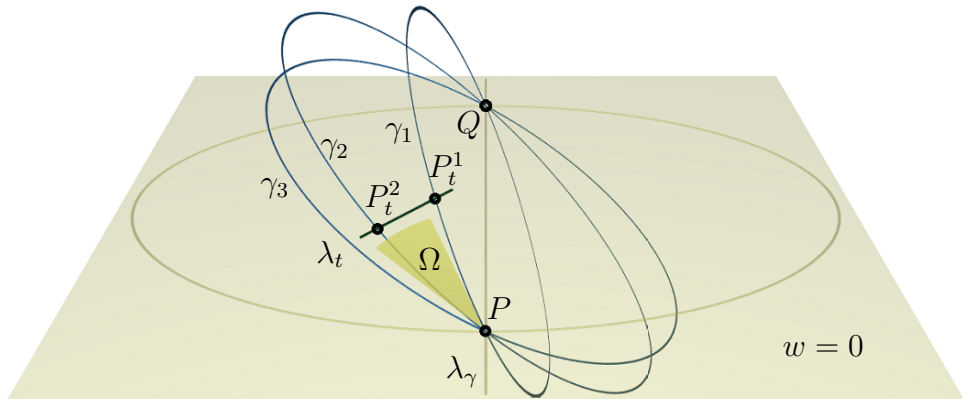


FIGURE 3. To the proof of Claim 2.7 case (3).

Case 3: *For some γ we have $\lambda_\gamma \cap I = I$; see Figure 3.* Then λ_γ is the infinitely distant line of the projective plane Π . Since the generating lines form an algebraic curve in $Gr(2, 4)$ it follows that there is a sequence of generating lines $\lambda_t \neq \lambda_\gamma$ converging to λ_γ . Since there are only finitely many generating lines crossing I , we may assume that $\lambda_t \cap I = \emptyset$. Take 3 pairwise noncoplanar generating circles γ_1, γ_2 , and γ_3 . By Claim 2.4 for each $i = 1, 2, 3$ the line λ_t intersects the circle γ_i at some

point P_t^i . Each of the 3 points P_t^i converges to one of the 2 points of the set I . By the pigeonhole principle we may assume that, say, P_t^1, P_t^2 converge to P . Then the plane $PP_t^1P_t^2$ converges to the projective plane Ω containing projective tangent lines to γ_1 and γ_2 at the point P (the tangent lines are distinct because γ_1 and γ_2 are not coplanar). The projective plane Ω has a unique common point with γ_1 while $\lambda_\gamma \subset \Omega$ intersects γ_1 by the 2-point set I . This contradiction proves the claim. \square

Proof of Lemma 2.6. By Claim 2.7 there are infinitely many generating circles γ such that $\bar{\Phi} \cap \Pi$ does not contain generating lines. Then $\bar{\Phi} \cap \Pi = \gamma$ because by Claim 2.4 the generating line through each point of $\bar{\Phi} \cap \Pi$ crosses γ . \square

Proof of Theorem 1.1. By Lemma 2.6 we have $\bar{\Phi} \cap \Pi = \gamma$ for infinitely many generating circles γ . So there is a generating circle γ with this property which is not a singular curve of the surface $\bar{\Phi}$, because $\bar{\Phi}$ contains only finitely many singular curves. By Claim 2.7 the plane Π does not touch the surface $\bar{\Phi}$ along the curve γ . Thus the circle γ has multiplicity 1 in the curve $\bar{\Phi} \cap \Pi$. By the Bezout theorem [5, Theorem 18.3] the degree of the surface $\bar{\Phi} \subset \mathbb{P}^3$ equals to the degree of its planar section (with multiplicity), and thus equals to 2. Since $\bar{\Phi}$ contains both real lines and real circles, it is either a one-sheeted hyperboloid, or a quadratic cone, or an elliptic cylinder. Theorem 1.1 is proved. \square

3. VARIATIONS

In this subsection we prove some related results. Throughout this section we work over the field of real numbers except otherwise explicitly indicated. First let us give a simple proof of the following theorem.

Theorem 3.1 (Takeuchi, 1987, [18]). *Let $\Phi \subset \mathbb{R}^3$ be a smooth closed surface homeomorphic to either a sphere or a torus. If through each point of the surface one can draw at least 7 distinct circles fully contained in the surface (and continuously depending on the point) then the surface is a round sphere.*

The proof just like the original one is topological. We simplify the original proof drastically using homology modulo 2 instead of integral homology.

Claim 3.2. *From any n smooth closed curves in the surface Φ intersecting pairwise in finitely many points one can choose greater or equal to $n/3$ curves intersecting pairwise in an even number of points (counted with multiplicities).*

Proof. If Φ is topologically a sphere then any two curves in Φ intersect in an even number of points (with multiplicities), and the claim follows. Assume that Φ is topologically a torus, and ξ and η are its “meridian” and “parallel” curves. Then the 1st homology group of Φ with coefficients modulo 2 consists of 4 homology classes: 0 , $[\xi]$, $[\eta]$, $[\xi] + [\eta]$. By the pigeonhole principle it follows that one of the 3 sets $\{0, [\xi]\}$, $\{0, [\eta]\}$, $\{0, [\xi] + [\eta]\}$ contains the homology classes of greater or equal to $n/3$ of the given n curves. Since $[\xi] \cap [\xi] = [\eta] \cap [\eta] = ([\xi] + [\eta]) \cap ([\xi] + [\eta]) = 0$ it follows that the curves from the same set intersect each other in an even number of points. \square

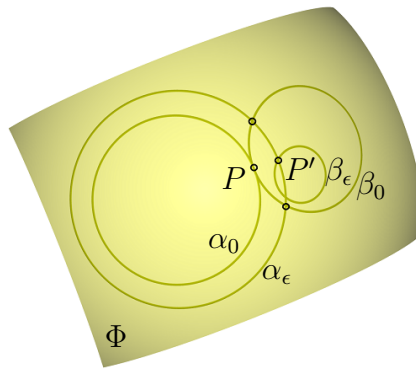


FIGURE 4. To the proof of Claim 3.3.

Claim 3.3. *Two circles passing through a generic point $P \in \Phi$ are transversal.*

Proof. Assume that the circles α_0 and β_0 passing through the point P touch each other; see Figure 4. Move the point P slightly along a geodesic in the normal direction to the circles. Draw the circles α_ϵ and β_ϵ through the resulting point P' of Φ . Then one of the pairs α_ϵ, β_0 or α_0, β_ϵ has 2 distinct intersection points near P . Hence there are points arbitrarily close to P such that the circles through them are transversal. \square

Proof of Theorem 3.1. The parity of the number of intersection points (with multiplicities) does not depend on the choice of particular curves from the families. Thus by Claim 3.2 there are 3 families $\alpha_t, \beta_t, \gamma_t$ of circles in Φ such that the circles from distinct families intersect in an even number of points (with multiplicities). Assume without loss of generality that the circles $\alpha_0, \beta_0, \gamma_0$ pass through a generic point $P \in \Phi$. Then by Claim 3.3 the circles $\alpha_0, \beta_0, \gamma_0$ intersect transversally.

Then for sufficiently small t any two of the circles $\alpha_0, \beta_0, \gamma_t$ have exactly two common points. Circles with 2 common points are cospherical. Thus α_0 and β_0 belong to one sphere. Each circle $\gamma_t \not\ni P$ belongs to the same sphere because it has at least 3 common points with the sphere (2 pairs of intersection points with α_0 and β_0 , minus at most 1 common point of the pairs). Since Φ is a smooth surface, it cannot jump from one sphere to another. Thus Φ is a sphere itself; cf. [17, Theorem 1]. \square

Now let us prove a new result generalizing an old one of Darboux.

Theorem 3.4. *Let $\Phi \subset \mathbb{R}^3$ be a smooth closed surface homeomorphic to either a sphere or a torus. If through each point of the surface one can draw at least 4 distinct circles fully contained in the surface (and continuously depending on the point) then the surface is a cyclide.*

Proof of Theorem 3.4. Let $\alpha_t, \beta_t, \gamma_t, \delta_t$ be the 4 families of circles in the surface Φ . Assume without loss of generality that the circles $\alpha_0, \beta_0, \gamma_0, \delta_0$ pass through a generic point P . Since Φ is topologically either a sphere or a torus by Claim 3.2 it follows that we can choose 2 families, say, α_t and β_t , such that for each s, t the circles α_t and β_s have an even number of intersection points (with multiplicities). By Claim 3.3 the circles α_0 and β_0 are transversal. Thus for s, t sufficiently close to zero the circles α_t and β_s are cospherical. Now the theorem follows from the following classical result. \square

Theorem 3.5. (See [4, Theorem 20 in p. 296]) *If through each point of a smooth surface in \mathbb{R}^3 one can draw 2 cospherical circular arcs fully contained in the surface (and continuously depending on the point) then the surface is a cyclide.*

Sketch of the proof. The surface is covered by two families of circular arcs α_t and β_t . Fix a sphere Σ orthogonal to both α_1 and α_2 . Take the spheres Σ_{1t} and Σ_{2t} containing $\alpha_1 \cup \beta_t$ and $\alpha_2 \cup \beta_t$, respectively. Since $\alpha_1 \perp \Sigma$ it follows that $\Sigma_{1t} \perp \Sigma$. Analogously, $\Sigma_{2t} \perp \Sigma$. If $\Sigma_{1t} = \Sigma_{2t}$ for each t then Φ is a sphere through the circles α_1 and α_2 . Otherwise $\beta_t \subset \Sigma_{1t} \cap \Sigma_{2t} \perp \Sigma$ for each t . Analogously $\alpha_t \perp \Sigma$ for each t . Assume without loss of generality that Σ is the unit sphere. Perform the *Darboux transformation* $(x, y, z) \mapsto 2(x, y, z)/(x^2 + y^2 + z^2 + 1)$ taking all circles orthogonal to Σ to line segments. The transformation takes our surface to a doubly ruled surface. Thus our surface is the preimage of a quadric under the Darboux transformation, i.e., a cyclide. \square

Now let us give examples showing nontriviality of our results. We begin with examples over the field of complex numbers.

Example 3.6. The irreducible complex cyclide $(x^2 + y^2 + z^2)^2 + (x + iy)^2 - z^2 = 0$, which can be parametrized in \mathbb{P}^3 as $t^2 - 1 : i(t^2 - 1 - 2st) : s(t^2 + 1) : s(t^2 - 1) + 4t$, is covered by a family of complex lines $t = \text{const}$ and a family of complex circles $s = \text{const}$ simultaneously.

Example 3.7. A general position degree 3 complex cyclide is covered by a family of complex circles and contains 27 complex lines; however, the surface contains no families of complex lines.

Proof. Any cyclide is covered by at least one family of complex circles [4, Chapter VII]. A general position degree 3 cyclide is nonsingular and hence contains exactly 27 complex lines. \square

Example 3.8. A general position ruled complex cubic surface is covered by a family of complex lines and contains 15 complex circles; however, the surface contains no families of complex circles.

Proof of Example 3.8. Consider the intersection I of the ruled cubic surface with the absolute conic. In general position it consists of six distinct points. Let P, Q be two of the intersection points. Let λ_1 be the line passing through P, Q , and let R be the third common point of λ_1 and the surface. In general position $R \neq P, Q$. Consider the ruling λ_2 passing through R . Take the plane Π containing the lines λ_1 and λ_2 . The intersection of Π and the surface consists of the ruling λ_2 and a curve γ of degree 2. The curve γ is irreducible once the plane Π contains neither the singular line nor the isolated line of the surface, i.e., the points P, Q, R do not belong to these lines (this follows from the classification of ruled cubic surfaces [11, Section 2]). Thus in general position γ is a conic through P and Q , i.e., a complex circle. There are 15 ways to choose two distinct points $P, Q \in I$ leading to 15 complex circles on the surface. \square

Finally, let us proceed to examples over the field of real numbers. Their obvious proofs are omitted.

Example 3.9. (See Figure 5 to the left.) The surface $(x^2 - z^2)(3z - 2) + (y - z)(3yz - 2y - 4z + 2) = 0$ is covered by a family of circles in the planes $z = \text{const}$ and contains 4 lines: $l_1(t) = (t, t, t)$, $l_2(t) = (-t, t, t)$, $l_3(t) = (t, 1 - t, 2t)$, $l_4(t) = (-t, 1 - t, 2t)$; however, the surface contains no families of lines.

Example 3.10. (See [14, Section 1] and Figure 5 in the middle.) The surface $(x^2 + y^2 + z^2 + 3)^2 - 4y^2z^2 - 16x^2 - 12y^2 = 0$ obtained by translation of a circle along another one is covered by 2 families of circles in the planes $y = \text{const}$ and $z = \text{const}$ but it is not a cyclide.

It is not known so far if there are surfaces covered by 3 families of circles besides cyclides. Let us give a related counterexample from *isotropic geometry* [13]. An *isotropic circle* is either a parabola with the axis parallel to Oz or an ellipse whose projection into the plane Oxy is a circle. An *isotropic cyclide* is given by equation (1), in which both instances of $(x^2 + y^2 + z^2)$ are replaced by $(x^2 + y^2)$ [14, Section 5].

Example 3.11. (See Figure 5 to the right.) The surface $z = xy(y - x)$ is covered by 3 families of isotropic circles in the planes $x = \text{const}$, $y = \text{const}$, $y - x = \text{const}$ but it is not an isotropic cyclide.

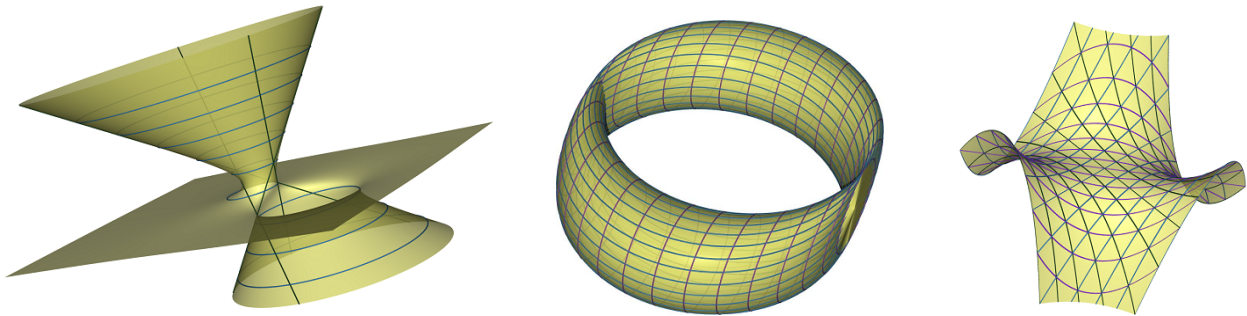


FIGURE 5. Left: a surface covered by 1 family of circles and containing 4 lines (Example 3.9). Middle: a surface covered by 2 families of circles (Example 3.10). Right: a surface covered by 3 families of isotropic circles (Example 3.11).

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